History of plasticity and metal forming analysis

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A B S T R A C T

The research history of mechanics, physics and metallurgy of plastic deformation, and the development of metal forming analysis are reviewed. The experimental observations of plastic deformation and metal forming started in France by Coulomb and Tresca. In the early 20th century, fundamental investigation into plasticity flourished in Germany under the leadership of Prandtl, but many researchers moved out to the USA and UK when Hitler came in power. In the second half of the 20th century, some analyzing methods of metal forming processes were developed and installed onto computers as software, and they are effectively used all over the world.

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1. Introduction

The phenomenon of plasticity has been studied from the view points of mechanics, physics and metallurgy, and many mathematicians contributed to refine the mechanics of plasticity. The research results are applied to geophysics and strength of materials, and of course, are used as indispensable tools for analyzing the metal forming processes.

Although the theories and the experimental results are explained in many text books, the building up process is not known well. In this article, the history of plasticity in relation to the analysis of metal forming is reviewed by putting emphasis on the personal profile of researchers.

2. Strength of materials and plasticity before the 20th century

2.1. Early days of strength of materials (Timoshenko, 1953)

Leonardo da Vinci (1452–1519) left many texts and sketches related to science and technology although he did not write books. One of the examples he studied is the strength of iron wire, on which hangs a basket being filled with sand. The strength of the wire can be determined by measuring the weight of sand when the wire is broken. Unfortunately, the idea and the advancement made by da Vinci were buried in his note, and were not noticed by scientists and engineers of the time.

It is generally accepted that Galileo Galilei (1564–1642) is the originator of modern mechanics. In his famous book “Two New Sciences”, he treated various problems related to mechanics, including an example of the strength of a stone beam. He organised his methods applicable to stress analysis into a logical sequence. His lecture delivered in the University of Padua attracted many scholars gathered from all over Europe, and disseminated the methods of modern science.

Robert Hooke (1635–1704) published the book “Of Spring” in 1678 showing that the degree of elongation of the spring is in proportion to the applied load for various cases. It is generally believed that Hooke came up with the idea of elastic deformation when he carried out experiments on the compressibility of air at Oxford University as an assistant of Robert Boyle (1627–1691), who put forward Boyle’s law.

2.2. Torsion test of iron wire by Coulomb (Bell, 1984)

In the paper submitted to the French Academy of Sciences in 1784, C.A. de Coulomb showed the results of torsion tests of iron wire carried out with the simple device given in Fig. 1. He estimated the elastic shearing modulus of the material from the frequency of torsional vibration, and measured the recovery angle after twisting. For a wire of length 243.6 mm and diameter 0.51 mm, the shearing elastic modulus was estimated to be about 8200 kgf/mm².

Fig. 2 shows the relation between the number of rotations in twisting and the angle of spring back. When the number of rota-
tions exceeds about 0.5, the angle of recovery becomes smaller than the angle of twisting, and the recovery angle increases only slightly when the number of rotation exceeds two times. This phenomenon suggests that plastic deformation starts on the surface of the wire when the rotation is about 0.5, and then the plastic zone expands towards the centre of the wire up to two rotations, and work-hardening proceeds gradually as the number of rotation increases further.

Let us estimate the yield stress and the flow stress of this wire from its dimensions and the elastic modulus. The shear strain and stress after 0.5 rotations are calculated respectively to be 0.003 and 24 kgf/mm², which is a little larger than the presently known shearing yield stress of iron, but may be reasonable if plastic deformation has already proceeded for 0.5 rotations. For the spring back angle of 45°, the shear stress is calculated to be $k = 50$ kgf/mm² if it is distributed uniformly across the cross-section. This shearing flow stress seems to be reasonable, too.

Charles A. de Coulomb (1736–1808) (Fig. 3) entered the military corps of engineers after receiving preliminary education in Paris. He was sent to the island of Martinique in the West Indies for 9 years. There he studied the mechanical properties of materials. In 1773, he submitted his first paper on the fracture of sandstone to the Academy. He concluded that fracture of sandstone occurred when the shear stress reached a certain value, similarly to the yield condition due to maximum shear stress.

After returning to France, he worked as an engineer, and continued to carry out research. In 1781, he won an Academy prize for his paper on friction, now known as Coulomb friction, and in the same year he was elected to membership of Academy.

2.3. Elasticity and stress–strain curve

In the early 19th century, the mathematical theory of elasticity began to flourish due to efforts of the scholars related with the École Polytechnique such as S.D. Poisson (1781–1840), Navier (1785–1836), A. Cauchy (1789–1857) and G. Lame (1795–1870), in parallel with those at the University of Cambridge University, being T. Young (1773–1829) and G. Green (1793–1841) (as reported by Timoshenko, 1953).

To determine the elastic constants, measurements of the stress–strain relations of metals were begun, and after extension in the elastic range, stress–strain curves in the plastic range were measured. Fig. 4 is the stress–strain curve of piano wire measured by F.J. Gerstner (1756–1832) and published in 1831 (Bell, 1984). He applied the load to a piano wire of 0.63 mm in diameter and 1.47 m in length with a series of weights. It is obvious that the plastic strain is measured after unloading.

2.4. H. Tresca (Bell, 1984)

H.E. Tresca carried out experiments on metal forming such as punching, extrusion and compression using various metals, and measured the relation between the forming load and ram displacement. He presented a series of papers to the French Academy of Sciences, starting in 1864. In Fig. 5, the cross-section of an extruded
billed made of 20 lead sheets is given. Tresca was interested in the metal flow as suggested by the title of his first paper, ‘Mémoire sur l’écoulement des corps solides à de fortes pression (On the flow of a solid body subjected to high pressure)’, rather than yielding in material testing.

Tresca assumed that the extrusion force $P$ could be expressed in terms of the shear stress $k$, and estimated the value of $k$ from the measured forming load of various processes. Because the values of shear stress $k$ estimated from the forming loads occurred in a certain range, he concluded that the metal flow occurred under a constant maximum shear stress. The values of shearing flow stress measured by Tresca are given in Table 1. It seems that the flow stress values are reasonable even from the present view point.

Henri E. Tresca (1814–1885) (Fig. 6) graduated from the École Polytechnique at the age of 19 in 1833, and sought a career in the design of civil structures. But his ambition was deterred by serious illness, and he spent many years teaching, building and performing tests on hydraulics. In 1852 he began to work at Conservatoire des Arts et Métiers in Paris as an engineer. He suddenly started research work when he was promoted to a major experimental physicist at the age of 50, and soon published many papers. After 8 years’ concentrated research activity, he was elected as a member of the French Academy of Sciences.

2.5. Saint-Venant and Lévy (Timoshenko, 1953)

When Tresca presented his paper to the French Academy of Sciences, Barré de Saint-Venant (1797–1886) was the authority of mechanics in France, elected a member of the Academy in 1868. After reading the experimental results of plastic flow by Tresca, his attention was drawn to the area of plasticity. In 1871, he wrote a paper on elastic–plastic analysis of partly plastic problems, such as the twisting of rods, bending of rectangular beams and pressurizing of hollow cylinders.

Saint-Venant assumed that (1) the volume of material does not change during plastic deformation, (2) the directions of principal strains coincide with those of the principal stresses (now known as total strain theory), and (3) the maximum shear stress at each point is equal to a specific constant in the plastic region.

The last assumption is now known as the Tresca yield criterion which is expressed with the maximum principal stress $\sigma_1$, the minimum principal stresses $\sigma_3$ and flow stress $Y$ as:

$$\sigma_1 - \sigma_3 = 2k = Y \quad (1)$$

Although his analyses are not complete from our current point of view, it can be said that plastic analysis started from this paper.

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**Table 1**

Shearing flow stress measured by Tresca.

<table>
<thead>
<tr>
<th>Material</th>
<th>Shearing flow stress (kgf/mm²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lead</td>
<td>1.82</td>
</tr>
<tr>
<td>Pure titanium</td>
<td>2.09</td>
</tr>
<tr>
<td>Lead–titanium alloy</td>
<td>3.39</td>
</tr>
<tr>
<td>Zinc</td>
<td>9.00</td>
</tr>
<tr>
<td>Copper</td>
<td>18.93</td>
</tr>
<tr>
<td>Iron</td>
<td>37.53</td>
</tr>
</tbody>
</table>
2.6. Bauschinger and Mohr (Timoshenko, 1953)

In the second half of the 19th century, the Technical Universities in German speaking areas became important research centres in plasticity and metal forming. They were established as PS: Polytechnische Schule and then changed to university level TH: Technische Hochschule.

Johann Bauschinger (1833–1893) graduated from Munich University and became a professor of Munich PS in 1868. He installed a 100 tons tension–compression universal testing machine with an extensometer of his own invention, and carried out a vast number of measurements of stress–strain relations. He found that the yield stress in compression after plastic tensile deformation was significantly lower than the initial yield stress in tension. Fig. 7 is the experimental result from tests carried out in 1885, in which the compression test is performed after a tension test up to a strain of 0.6%. It is seen that the initial yield stress was 20.91 kgf/mm² and the yield stress in compression after tensile deformation was 9.84 kgf/mm².

In 1882, Otto Mohr (1835–1913) presented a graphical representation of stress at a point. On a graph with axes indicating normal and shear stress components, the stress state of a point on a plane is expressed by a circle. Mohr used his representation of stress to devise a strength theory.

Fig. 8 shows the stress circles for cast iron tested in tension, compression, and in torsion. Mohr suggested that the envelope of the circles was a fracture limit. This idea was extended to a yield condition in which shearing yield stress was affected by hydrostatic pressure. This condition is often called “Mohr’s yield condition”.

Mohr graduated from Hannover PS and worked as a structural engineer. When he was 32 years old, he was already a well-known engineer and was invited by Stuttgart TH. After teaching engineering mechanics there until 1873, he moved to Dresden TH and continued teaching.

2.7. J. Guest (Guest, 1900)

In 1900, James Guest (University College London) published a paper from the Royal Society on the strength of ductile materials under combined stress states. By carrying out tension and torsion tests of internally pressurized tubes, he examined the occurrence of yielding. Guest is the first person to differentiate yielding of ductile metal from brittle fracture, where previously ‘failure’ had been used to express the strength limit of material both due to yielding and brittle fracture.

He came to the conclusion that yielding occurs when the maximum shear stress reaches a certain value. In Fig. 9, the yielding points are plotted on a graph of principal stresses in plane stress. Although his conclusion was the same as that of H. Tresca, he naturally thought that he had found a new criterion of yielding because he did not recognize that the large plastic flow observed by Tresca and the initial yielding he observed were essentially the same phenomenon.

3. Yield criteria and constitutive equations

3.1. Progress of research in yield condition

During the 19th century, the maximum shear stress criterion was established by Tresca, Saint-Venant, Mohr and Guest. The yield criterion of elastic shear-strain energy, mostly called Mises yield criterion, was put forward in the early 20th century. It is written by the following equation with the maximum, medium and minimum principal stresses $\sigma_1 \geq \sigma_2 \geq \sigma_3$ as:

$$\frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = Y$$

In 1904, M.T. Huber proposed this criterion (Engel, 1994) although limited to compressive hydrostatic stress conditions. This paper was not known for 20 years by the researchers of plasticity.
because it was written in the Polish language. von Mises (1913) wrote his paper from the view point of mathematics without discussing the physical background. Hencky (1924) introduced Hube’s paper and derived the criterion of elastic–shear strain–energy. Nádai (1937) showed that the criterion could be interpreted as a prediction that yielding occurs when the shear stress on the octahedral plane in the space of principal stresses reaches a critical value. This idea is now often used in textbooks of plasticity because of its simple graphical representation, although it has no physical meaning.

The difference between the Tresca yield criterion Eq. (1) and the Mises criterion Eq. (2) when expressed with principal stresses is that the medium principal stress (σ₂) has an effect (Mises) or does not (Tresca). This difference was experimentally examined by Lode (1926) following a suggestion by Nádai, and then Taylor and Quinney (1932). Their conclusions were that the medium principal stress did influence the yielding condition for mild steel, aluminium and copper and thus the Mises criterion offered a better approximation for the yield condition.

Hill (1948) proposed a yield criterion for anisotropic materials, and since then many researchers have also tried to express the yielding behaviour of anisotropic materials. Yield criteria for other materials which are not incompressible and isotropic have also been put forward: one example is the criterion for porous or compressible metals proposed by Shima and Oyane (1976).

### 3.2. Constitutive equation

In his paper published in 1872, M. Lévy (according to Timoshenko, 1953) used an incremental constitutive equation. von Mises proposed the same constitutive equation because Lévy’s paper was not known outside of France. Mises considered that the increments of plastic strain components \( \Delta e_1^p, \Delta e_2^p, \Delta e_3^p \) were in proportion to the deviatoric stress components \( \sigma_1', \sigma_2', \sigma_3' \), where for example \( \sigma_1' = \sigma_1 - (\sigma_1 + \sigma_2 + \sigma_3) / 3 \). Thus,

\[
\frac{\Delta e_1^p}{\sigma_1'} = \frac{\Delta e_2^p}{\sigma_2'} = \frac{\Delta e_3^p}{\sigma_3'}
\]  

(3)

In the plastic strain range of an elastic–plastic material, the increments of elastic strain components \( \Delta e_1^e, \Delta e_2^e, \Delta e_3^e \), and the plastic strain increments \( \Delta e_1^p, \Delta e_2^p, \Delta e_3^p \), should be handled separately. Prandtl (1924) treated this problem for plane-strain, and Reuss (1930) (Budapest Technical University) showed the expression for all of the strain components. For example,

\[
\Delta e_1 = \Delta e_1^e + \Delta e_1^p = \frac{1}{E} \left\{ \Delta \sigma_1 - \nu (\Delta \sigma_2 + \Delta \sigma_3) \right\} \\
+ \frac{\nu}{Y} \left\{ \sigma_1 - \frac{1}{2}(\sigma_2 + \sigma_3) \right\}
\]

(4)

where \( \Delta \bar{e} \) is an equivalent strain increment which is expressed in terms of the plastic strain increments, and \( Y \) is the flow stress.

In the 1960s, when the finite element analysis of elastic–plastic material was under development, a key topic was the inversion of the above Prandtl–Reuss equation to express the stress increments in terms of strain increments. However it was eventually found that R. Hill had already done this work in his book published in 1950 (Hill, 1950).

### 3.3. Letter of J. Maxwell (Timoshenko, 1953)

The letters from James Clerk Maxwell (1831–1879; famous for Maxwell’s equation) to his friend William Thomson (Lord Kelvin: 1824–1907) were published in 1937 in a book, and it was then found that Maxwell had written about the occurrence of yielding as early as 1856.

Maxwell showed that the total strain energy per unit volume could be resolved into two parts (1) the strain energy of uniform tension or compression and (2) the strain energy of distortion. The total elastic energy per unit volume is expressed as:

\[
F = \frac{1}{2} \frac{E}{3(1-2\nu)} (\sigma_1 + \sigma_2 + \sigma_3)^2 \\
+ \frac{1}{2} \frac{E}{3(1+2\nu)} \left\{ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right\}
\]

(5)

where \( E \) is Young’s modulus, \( \nu \) is Poisson’s ratio and, \( \sigma_1, \sigma_2, \sigma_3 \) are principal stresses. The first term on the right side of the equation is the energy for volume change due to uniform tension or compression, and the second term is the energy of distortion.

Maxwell made the statement: “I have strong reasons for believing that when the strain energy of distortion reaches a certain limit, then the element will begin to give way.” Further on the states: “This is the first time that I have put pen to paper on this subject. I have never seen any investigation of the question, given the mechanical strain in three directions on an element, when will it give way?” Unfortunately he did not return to this subject again.

### 3.4. M.T. Huber (Engel, 1994; Olesiak, 2000)

In 1904, M.T. Huber proposed that yielding was determined by elastic shear-strain energy distortion when the hydrostatic stress was compressive, and by the total elastic energy when the hydrostatic stress was tensile. Huber’s paper in the Polish language did not attract general attention until H. Hencky introduced it in 1924.

Maksymilian Tytus Huber (1872–1950) (Fig. 10) graduated from Lwów (now Lviv, Ukraine) Technical University in 1895, and stud-
ied mathematics at Berlin University. In 1899 he began to work at a technical school in Kraków and wrote his paper on the yield condition. In 1909 he was invited to head the Chair of Technical Mechanics at Lvów Technical University. When World War I began, he was called to the Austria–Hungary army but captured by Russian troops. After World War I, he went back to Lvów Technical University and became the president of the university. In 1928 he moved to Warsaw Technical University, and actively took part in various advisory and expert bodies. He became a member of the Polish Academy of Learning. During World War II, he was not able to work in the university and lost all his belongings in a fire ignited by Germans troops during the Warsaw uprising of 1944, but after the war he was able to continue research work in Gdansk Technical University.

3.5. R. von Mises

von Mises (1913) manipulated the maximum shear stresses on the principal stress planes:
\[
\tau_1 = \frac{\sigma_3 - \sigma_2}{2}, \quad \tau_2 = \frac{\sigma_1 - \sigma_3}{2}, \quad \tau_3 = \frac{\sigma_2 - \sigma_1}{2}
\]  
(6)

It is apparent that simple summation of the maximum shear stresses is always zero:
\[
\tau_1 + \tau_2 + \tau_3 = 0
\]  
(7)

In the space of the maximum shear stresses, he expressed the criterion of maximum shear stress:
\[
|\tau_1| \leq k, \quad |\tau_2| \leq k, \quad |\tau_3| \leq k
\]  
(8)

which Mises called Mohr’s yield criterion, and is represented in the cube shown in Fig. 11. The yield condition of maximum shear stress is expressed as the intersection of the cube and Eq.(7) is given by the hexagon in the figure.

Then he considered a sum of squares of the shear stresses as the sphere in the figure.
\[
k^2 = \tau_1^2 + \tau_2^2 + \tau_3^2 = \frac{1}{4} \left\{ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right\}
\]  
(9)

The circle resulted as the intersection of the sphere and Eq. (7) is an approximation of the Mohr (Tresca) criterion. While the Mohr criterion cannot be expressed by a simple mathematical equation, the new criterion is easy to handle mathematically, as is often done in mathematical plasticity.

Richard von Mises (1883–1953) (Fig. 12) was born in Lemberg (now Lviv, Ukraine) and graduated in mathematics from the Vienna University of Technology. In 1908 Mises was awarded his doctorate from Vienna. In 1909, at the age of 26, he was appointed professor in Straßburg (now Strasbourg, France) and received Prussian citizenship. There he wrote his paper on the yield criterion. During World War I, he joined the Austro–Hungarian army and flew as a test pilot, and then supervised the construction of a 600HP aircraft for the Austrian army.

After the war Mises held the new chair at Dresden TH. In 1919, he was appointed director of the new Institute of Applied Mathematics created in the University of Berlin. In 1921 he became the editor of the newly founded journal “Zeitschrift für Angewandte Mathematik und Mechanik” and stayed until 1933. With the rise of the Nazi party to power in 1933, Mises felt his position threatened. He moved to Turkey, where he held the newly created chair of Pure and Applied Mathematics at the University of Istanbul. In 1939, amid political uncertainty in Turkey, he went to the USA, where he was appointed in 1944 as the Gordon-McKay Professor of Aerodynamics and Applied Mathematics at Harvard University.

3.6. A.L. Nádai

Arpad L. Nádai (1883–1963) (Fig. 13) was born in Hungary and graduated from Budapest University of Technology, and then studied in Berlin TH getting his doctorate in 1911. In 1918 he moved to L. Prandtl’s Institute of Applied Mechanics in Göttingen and was promoted to professor in 1923. In 1927, he moved to the Westinghouse Laboratory in the USA as the successor of P.E. Timoshenko. Thus his paper in 1937 on the yield criterion was based on work performed in the USA.

In 1927, Nádai published a book of plasticity in German and this was translated into English as “Plasticity – A Mechanics of the Plastic State of Matter” (Nádai, 1931) as the first English book of plasticity. The characteristic feature of this book is that it consists of two parts, (1) plasticity of metals and (2) application of plasticity in geophysics problems. In 1950 the first part of this book was rewritten and published as “Theory of Flow and Fracture of Solids.”

Fig. 11. Handling of maximum shear stresses by Mises.

Fig. 12. R. von Mises.
4. Physics and metallurgy of plastic deformation

4.1. Plastic deformation of a single crystal

In 1923, P.W. Bridgman invented a method to make a single crystal of metal by pulling it out of molten metal. Since M. von Laue (1879–1960) had already established the method for determining the direction of the crystal lattice by X-ray diffraction, the study of plastic deformation of single crystals could immediately begin. Taylor and Elam (1923) carried out a tension test of an Al single crystal (Fig. 14), and found that plastic deformation occurred by sliding on a certain crystallographic (sliding) plane in a definite (sliding) direction, and the critical shear stress \( \tau_{cr} \) on the plane was calculated. They continued experiments with single crystals of iron, gold, copper and brass. In Germany, the groups of Schmid (1926) and Göler and Sachs (1927) presented their results from tension tests of single crystals. Sachs (1928) calculated the yield stress of polycrystalline metal as an average of those of single crystals with random orientations. The calculated average yield stresses in tension and shearing were \( \sigma_y = 2.24 \tau_{cr} \) and \( \sigma_f = 1.29 \tau_{cr} \), respectively, and the ratio was \( \sigma_y/\sigma_f = 0.577 \). This ratio was the same as that derived from Mises yield condition, but the calculated constant 2.24 was too small compared with the experimental value. Taylor (1938) proposed a method to relate the yield stress of polycrystalline metals with that of single crystals by taking account of the constraints provided by neighbouring grains, giving \( \sigma_y = 3.96 \tau_{cr} \), which was quite near to the value obtained by experiments.

When a single crystal is plastically stretched, the direction of the crystal rotates as demonstrated in Fig. 15 due to sliding over the specific planes, and the sliding planes tend to become parallel to the stretching direction irrespective of the initial orientation. This means that anisotropy is developed by plastic deformation of polycrystalline metals. Boas and Schmid (1930) were the first to study the development of anisotropy.

4.2. Dislocation theory

When the initially polished surfaces of single crystals were observed after plastic deformation, slip bands (Fig. 16) were observed suggesting that sliding occurred on a limited number of sliding planes. Since an extremely large shear stress, 1000–10,000 times as large as the measured critical shear stress, would be needed to overcome the atomic bonding stress, many researchers worked to understand the mechanism of plastic deformation. Taylor (1934), Polanyi (1934), and Orowan (1934) independently proposed the sliding mechanism by crystal defects, i.e. dislocations. Fig. 17 shows the explanation by Taylor about dislocations in a crystal lattice during plastic deformation. The existence of dislocations was proved in the 1950s after electronic microscopy was invented.

When the general assembly of the International Union of Physics was held in Tokyo in 1953, N.F. Mott (1905–1999), the president of the Union, told the delegates that the first person to recognize the existence of dislocations was K. Yamaguchi. Yamaguchi (1929) showed a representation of dislocations as shown in Fig. 18 to explain the cause of the warping of a single crystal after plastic deformation. Yamaguchi carried out the research in the Institute of Physics and Chemistry at the laboratory of M. Mashima, who had a close relation with the laboratory of G. Sachs in Germany. In 1937, Yamaguchi was appointed a professor of Osaka Imperial University, when it was established.
4.3. Response of metals to high deforming speed

The measurement of stress–strain curves began to attract researchers in the second half of the 19th century but appropriate measuring methods for high speed phenomena did not exist. Dunn (1897) carried out compression tests by using a drop hammer, and measured the displacement of the hammer optically and recorded it on a film attached to a rotating drum. By differentiating the displacement, he calculated the hammer velocity, and then obtained the acceleration or force applied to the hammer by differentiating the velocity. From the measured displacement and the calculated force, he was able to determine the stress–strain curves. In the early 20th century, high speed stress–strain curves were obtained by some groups in Europe with similar measuring methods.

Itihara (1933) (Tohoku Imperial University) measured the shearing stress–shearing strain curve at high speed and high temperature up to 1000 °C. Fig. 19 shows the equipment of the torsion test in which the torque was determined by the twisting angle of the measuring bar.

Manjoine and Nádai (1940) measured the stress–strain curves in a high speed tension test at up to 1000 °C as shown in Fig. 20 by using a load cell with strain gauges.

Kolsky (1949) used the split Hopkinson bar, which was developed by B. Hopkinson at the University of Cambridge in 1914 (Bell, 1984). To measure a high strain rate stress–strain curve, a specimen was sandwiched between two long bars, one end of which was struck and the transmitted elastic wave was measured at the other.

4.4. P.W. Bridgman (Bridgman, 1964)

Although von Kármán carried out compression tests on marble under high pressure and published the results in 1911, the mechanical behaviour of metals under high hydrostatic pressure was mainly studied by P.W. Bridgman during the first half of the 20th century. Although he found that the ductility of metal was...
remarkably enhanced by pressure as Fig. 21, he was more interested in the effect of pressure on the stress, which is only a little affected by pressure as shown in Fig. 22.

Percy Williams Bridgman (1882–1961) (Fig. 23) studied physics in Harvard University and received his Ph.D. in 1908. He was appointed Instructor (1910), Assistant Professor (1919), before becoming Hollis Professor of Mathematics and Natural Philosophy in 1926. He was appointed Higgins University Professor in 1950. From 1905, Bridgman continued the experiments under high pressure for about 50 years, and published the results on plastic deformation of metals in the book “Studies in Large Plastic Flow and Fracture”. He invented a method for growing single crystals and proposed a calculation method for the stress state in the neck of a tensile test specimen. A machinery malfunction led him to modify his pressure apparatus; the result was a new device enabling him to create pressures eventually exceeding 100,000 kgf/cm² (10 GPa). This new apparatus brought about a plenty of new findings, including the effect of pressure on electrical resistance, and on the liquid and solid states. In 1946, he received the Nobel Prize in physics for his work on high pressure physics.

4.5. G.I. Taylor

Geoffrey Ingram Taylor (1886–1975) (Fig. 24) was born in London, and studied mathematics at Trinity College, Cambridge. At the outbreak of World War I he was sent to the Royal Aircraft Factory at Farnborough to apply his knowledge to aircraft design.
After the war, Taylor returned to Trinity working on an application of turbulent flow to oceanography. In 1923, he was appointed to a Royal Society research professorship as a Yarrow Research Professor. This enabled him to stop teaching which he had been doing for the previous 4 years and which he both disliked and for which he had no great aptitude. It was in this period that he did his most wide ranging work on the deformation of crystalline materials which led on from his war work at Farnborough.

During World War II Taylor again worked on applications of his expertise to military problems. Taylor was sent to the United States as part of the British delegation to the Manhattan project. Taylor continued his research after the end of the War serving on the Aeronautical Research Committee and working on the development of supersonic aircraft. Though technically retiring in 1952, he continued researching for the next 20 years.

4.6. M. Polanyi

Michael Polanyi (1891–1976) was born into a Jewish family in Budapest, Hungary and graduated from medical school of Budapest University. His scientific interests led him to further study in chemistry at the Karlsruhe TH in Germany and he was awarded his doctorate in 1917. In 1920, he was appointed a member of the Kaiser Wilhelm Institute for Fibre Chemistry, Berlin, where he developed new methods of X-ray analysis and he made contributions to crystallography including dislocation theory.

In 1933, he resigned his position in Germany when Hitler came to power. Within a few months he was invited to take the chair of physical chemistry at the University of Manchester in England. He believed from his experience in science that there was a necessary connection between the premises of a free society and the discovery of scientific truths. In 1938 he formed the Society for the Freedom of Science.

4.7. E. Orowan

Egon Orowan (1902–1989) (Fig. 25) was born in Budapest and received his doctorate from Berlin TH on the fracture of mica in 1932. He had difficulty in finding employment and spent the next few years ruminating on his doctoral research, and completed his paper on dislocations.

After working for a short while on the extraction of krypton from the air for the manufacture of light bulbs in Hungary, Orowan moved in 1937 to the University of Birmingham where he worked on the theory of fatigue collaborating with R. Peierls (1907–1955).

In 1939, he moved to the University of Cambridge where W.L. Bragg (1890–1971: X-ray analysis) inspired his interest in X-ray diffraction. During World War II, he worked on problems in munitions production, particularly developing an understanding of plastic flow during rolling. In 1950, he moved to MIT where, in addition to continuing his metallurgical work, he developed his interests in geological and glaciological deformation and fracture.

5. Slip-line field method

5.1. Progress of the slip-line field method

Prandtl (1920) presented an analysis of the plane-strain indentation of a flat punch into a rigidplastic solid body as shown in Fig. 26. He assumed a rigid-perfectly plastic material without work-hardening but with a pressure sensitive flow stress (Mohr yield criterion). By solving the equilibrium equation, he constructed a series of lines having directions parallel to the maximum shear stress as shown in Fig. 26. He correctly obtained the indenting pressure for a material with shearing flow stress \( k \) without pressure sensitivity as:

\[
p = 2k \left(1 + \frac{n}{2}ight)
\]

Hencky (1923) derived a general theorem of stress states for slip-line fields which now carries his name. A statically admissible stress field which satisfies the equilibrium equation, yield condition and boundary force is not always correct, because the velocity field associated with the stress state may not satisfy volume constancy or may lead to negative energy consumption. Geiringer (1930) derived an equation in relation to the velocity field by considering the incompressibility condition in plastic deformation and the relation between strain rate and velocity.

In 1933, when the fundamentals of slip-line field theory were established, the Nazis came to power in Germany and forced the Jewish researchers to leave from university positions. The remarkable progress attained in the field of plasticity was thus halted in Germany. The researchers expelled from Germany tried to find a safe haven in Turkey, the United States and England, where research work on plasticity was replanted.

During World War II, R. Hill used the slip-line field method to predict the plastic deformation of a thick plate being penetrated by a bullet. He proposed slip-line fields for various problems such as wedge indentation, compression of thin plates with friction, plate drawing (Fig. 27) and tension tests for a notched plate. He used slip-
line theory which had been developed as a mathematical method for the purpose of practical engineering purposes.

Sokolovskii (1948) reported that active research works were performed in the area of slip-line fields in the Soviet Union too. This is possibly due to the influence of H. Hencky, who established the slip-line theory and stayed in the Soviet Union to carry out research until 1938.

Hill (1950), and Prager and Hodge (1951), first presented a systematic account of slip-line theory and displayed the engineering worth of the approach. Prager’s introduction of the hodograph, or velocity plane diagram, in 1953 introduced a vast simplification into the handling of slip-line solutions and removed the difficulties.

During the 1950s and 60s, many new slip-line fields were proposed (Johnson et al., 1970) for extrusion, rolling, drawing and metal cutting. Since the slip-line method was the only mean to allow prediction of the stress state in the deforming material at that time, it was used widely even though the required assumption of plane-strain behaviour was not realistic for most metal forming operations. When finite element methods enabled precise stress calculation in axi-symmetric and later 3D problems, the use of the slip-line field method decreased from around 1980, although its academic value was not lost.

5.2. L. Prandtl

Ludwig Prandtl (1870–1953) (Fig. 28) received his engineering education at the Munich TH. After graduating, he remained at the school as an assistant of A. Föppl (1854–1924: successor of J. Bauschinger), and carried out doctoral work on the bending of circular plates. After working in industry for a while, he was appointed as a professor of industrial mechanics at Hannover TH in 1900. There he proposed the membrane analogy of torsion and the boundary layer of fluid flow. In 1904 he was invited to the Institute of Mechanics in Göttingen University. Soon he began to study plasticity such as plastic buckling and bending. He was appointed the leader of the laboratory of aerodynamics, and studied wing theorems and other important works of fluid dynamics.

In 1922, Prandtl established the society of applied mathematics and mechanics, “Gesellschaft für Angewandte Mathematik und Mechanik”, and led the area of applied mechanics. He is also famous as the teacher of many leaders in mechanics in the 20th century such as Th. von Kármán (California Institute of Technology), S.P. Timoshenko (Stanford University), A. Nádai (Westinghouse Laboratory), W. Prager (Brown University) and others.

5.3. H. Hencky

Heinrich Hencky (1885–1951) (Fig. 29) graduated from Darmstadt TH and began to work in Ukraine as an engineer of a railway company in 1913 at the age of 28. Soon World War I began and the area was occupied by Russian troops, and he was kept in a camp in Ural, where he married a Russian woman.

Although he could not find a permanent job after the war in Germany, he was awarded his Habilitation (qualification for professorship) from Dresden TH and found a job in Delft Technical University in 1922. He carried out research into slip-line field theory in Delft, and stayed until 1929.

In 1930, he moved to MIT in the USA, but his scientific approach to engineering was not accepted there because practical technologies were overwhelming, and he resigned from MIT after only 2 years. In 1936, Hencky was invited to the Soviet Union by B.G. Galerkin (1871–1945: variational method) and carried out research in Moscow University. But in 1938, as relations between the Soviet Union and Germany worsened, he returned to Germany, and worked in a bus manufacturing company in Mainz.

Fig. 27. Slip-line fields for drawing and extrusion proposed by Hill (1950).

Fig. 28. L. Prandtl.
5.4. H. Geiringer

Hilda Geiringer (1893–1973) (Fig. 30) was born in Vienna and received her doctorate in 1917 from the University of Vienna with a thesis about Fourier series. From 1921 to 1927 she worked at the Institute of Applied Mathematics in the University of Berlin as an assistant of von Mises. Her mathematical interests had switched from pure mathematics to probability and the mathematical development of plasticity theory. In 1927, Geiringer became Privatdozent (lecturer). During this period she had a brief marriage and had one daughter.

In 1930, her work in plasticity theory led to the development of the fundamental Geiringer equations for plane-strain plastic deformations. Geiringer remained at the University of Berlin until forced to leave when Hitler came to power. After a brief stay as a research associate at the Institute of Mechanics in Belgium, she became a professor of mathematics at Istanbul University in Turkey where she stayed for 5 years.

In 1939, she emigrated to the United States with the help of A. Einstein, and became a lecturer at Bryn Mawr College. While at Bryn Mawr she married R. von Mises who was then teaching at Harvard. In 1944, Geiringer became professor and chair of the mathematics department at Wheaton College in Massachusetts. Attempts to find a position at some of the larger universities near Boston repeatedly failed, often because of her gender. From 1955 to 1959, she worked as a research fellow in mathematics at Harvard in addition to her position at Wheaton to complete her husband’s unpublished manuscripts “Mathematical Theory of Probability and Statistics” after his death in 1953. Geiringer was elected a fellow of the American Academy of Arts and Science.

5.5. W. Prager (Hopkins, 1980)

William Prager (1903–1980) (Fig. 31) was born in Karlsruhe, and studied at Darmstadt TH receiving his doctorate in 1926 at the age of 23. From 1929 to 1933 he worked as the acting director of Prandtl’s Applied Mechanics Institute at Göttingen. At the age of 29, he was appointed at Karlsruhe TH as the youngest professor in Germany, but soon he was dismissed when Hitler came to power. He was invited to Istanbul University, Turkey and acted as a special adviser in education to the government. Prager remained in Istanbul until 1941. The expansion of World War II made the position
of the German refugees insecure, and he accepted the invitation of Brown University in the USA made on the recommendation of A. Einstein.

The Graduate Division of Applied Mathematics at Brown University was created in 1946 with Prager as its first Chairman, a position he held until 1953. By his effort, Brown University became the centre of applied mechanics, especially in the area of plasticity in the 1950s and 60s.

5.6. R. Hill (Hopkins and Sewell, 1982)

Rodney Hill (1921–) (Fig. 32) was born in Yorkshire, England and read mathematics at Gonville and Caius College, Cambridge, where E. Orowan was teaching. In 1943, amid World War II, he joined a theoretical group on armaments led by N. Mott and he was assigned the problem of deep penetration of thick armour by high-velocity shells. This aroused Hill’s interest in the field of plasticity. From 1946, he began to work with the group of metal physicists under E. Orowan at the Cavendish Laboratory in Cambridge. He solved various metal forming problems using plasticity theory, and obtained his Ph.D. in 1948. In 1949, he was invited to be the head of a new Section of the Metal Flow Research Laboratory of the British Iron and Steel Association (BISRA).

Hill expanded his Ph.D. thesis and published the book “The Mathematical Theory of Plasticity” in 1950 when he was 29 years old. This book was soon accepted as a standard of mechanics of plasticity. In 1952, he became the Editor in Chief of a new journal, the “Journal of Mechanics and Physics of Solids”, which was eventually known as the highest level journal in mechanics. In 1953 he applied for and was offered the post of a new Chair of Applied Mathematics in Nottingham University, and undertook administrative work on top of the research works of plasticity until his retirement from the university in 1962. From 1963, Hill moved back to Cambridge and continued research work in solid mechanics.

6. Slab method

6.1. Analysis of forging by E. Siebel

Siebel (1923) wrote a paper on the analysis of forging. He assumed a thin area (slab) on which to define an equilibrium equation. In the case of the compression of a cylinder of diameter \(d\) and height \(2h\) shown in Fig. 33, he considered a thin layer of a thickness \(dx\) and a height the same as the cylinder, and defined an equilibrium of forces acting on the layer.

For the case of the yield stress \(Y\) and friction coefficient \(\mu\), he derived the average contacting pressure \(\bar{P}\) as:

\[
\bar{P} = \frac{Y}{\mu d} \left( 1 + \frac{1}{3} \mu \left( \frac{d}{h} \right) \right)
\]

(11)

Using more recent applications of the slab method, the average pressure is calculated to be

\[
\bar{P} = 2Y \left( \frac{h}{\mu d} \right)^2 \left\{ \exp \left( \frac{\mu d}{2h} \right) - \frac{\mu d}{2h} - 1 \right\} \equiv Y \left( 1 + \frac{1}{3} \mu \left( \frac{d}{h} \right) \right)
\]

(12)

The last equation, identical to Siebel’s solution, is an approximation of the second equation when \(\mu d/h\) is sufficiently smaller than 1.0. Siebel numerically calculated the average pressures for some typical cases of forging, and discussed ways to apply the result to backward extrusion.

Soon after Siebel’s paper, similar methods were used by von Kármán (1925) for analyzing rolling of sheet metal and by Sachs (1927) for solving wire drawing. Using this method, Siebel continued to analyze various processes, and many researchers extended the method. Since the results are mathematically analytical, they are widely used in industry (Lippmann and Mahrenholts, 1967; Lange, 1985).

Erich Siebel (1891–1961) (Fig. 34) received his doctorate from Berlin TH in 1923 at the age of 32, on the topic of the calculation of load and energy in forging and rolling. After working in the steel industry for a short time, he became a leader of the metal forming division at Kaiser Institute in Dusseldorf (steel), and carried out analysis of rolling and forging. In 1931, he became a professor of Stuttgart TH, and treated almost all areas of metal forming such as deep drawing and wire drawing. He continued theoretical research into metal forming until retirement in 1957.
6.2. Th. von Kármán (von Kármán, 1967)

In 1925, Th. von Kármán presented a short paper on rolling to a meeting of applied mechanics. In three pages, the fundamental differential equation, a result of pressure distribution and energy efficiency were given. Although this paper gave a great influence to the subsequent researches in rolling technology, Kármán never returned to this subject again.

Theodore von Kármán (1881–1963) (Fig. 35) was born into a Jewish family in Budapest, Austria – Hungary and he studied engineering at the Royal Technical University in Budapest. After graduating in 1902, he joined Prandtl’s Institute at Göttingen University, and received his doctorate in 1908. Then he taught at Göttingen for 4 years. During this period he measured stress–strain curves for marble under high pressures. He was also interested in vibration induced by fluid flow and found the Kármán vortex, which made him famous in the area of fluid dynamics.

In 1912, he accepted a position as director of the Aeronautical Institute at RWTH Aachen. His time at RWTH Aachen was interrupted by service in the Austro–Hungarian Army 1915–1918, where he designed an early helicopter. In his own biography, he does not mention his work on rolling analysis in 1925, possibly because his main interest was building a wind tunnel in Aachen.

In 1927, Kármán stayed in Japan for a while as an advisor to an airplane company to build a wind tunnel. In 1930, he accepted the directorship of the Guggenheim Aeronautical Laboratory at the California Institute of Technology (Caltech) and emigrated to the United States. He is one of the founders of the Jet Propulsion Laboratory, which is now managed and operated by Caltech. In 1946 he became the first chairman of the Scientific Advisory Group which studied aeronautical technologies for the United States Army Air Forces. At age 81 von Kármán received the first National Medal of Science, bestowed in a White House ceremony by President John F. Kennedy.

6.3. Development of rolling analysis (Lippmann and Mahrenholts, 1967)

For the case of flat rolling shown in Fig. 36, von Kármán (1925) derived the equilibrium equation for the position x, plate thickness h and roll angle θ, as:

\[ d\left(\frac{h \cdot q}{2}\right) = p \left(\tan \theta \mp \tan f\right) dx \]  

(13)

where q is the horizontal pressure acting within the plate, p is the pressure acting on the roll surface, and the friction between the plate and the roll is \( \mu = \tan f \). The minus (−) sign in the equation is for the entrance side and the (+) sign is for the exit side.

This equation is valid when the plate thickness is small and the stress in the thickness direction is almost constant, and further, friction is described by Coulomb’s law, as is the case for cold rolling of thin plate.

Tresca’s yield condition Eq. (1) is written as:

\[ p_1 - q = Y \]  

(14)

where \( p_1 \) is the vertical pressure (perpendicular to q), which can be calculated from the roll pressure p and the frictional stress \( \mu p \) as:

\[ p_1 = p \pm \mu p \tan \theta \quad (+ \text{for entrance side}) \]  

(15)

Since the parameters x, h and θ are related to each other when the roll radius is given, it is necessary to use only one of them in solving the differential equation. The stresses p, q and \( p_1 \) are not independent, and one of them should be chosen as the variable. Thus the equation to be solved may be in the form of

\[ \left(\frac{p}{q}/p_1\right) = f \left(x/h/\theta\right) \]  

(16)

but the result of this problem cannot be presented explicitly.
The distribution of roll pressure in Fig. 37 by Kármán was obtained numerically, but for designing the rolling mills it is necessary to present the rolling torque, roll force and the energy explicitly. By introducing various approximations, Trinks (1937), Nádai (1939) and Hill (1950) made approximate mathematical expressions leading to nomograms to be used in industry.

In applying theoretical results to cold rolling, roll flattening cannot be neglected because the roll is elastically deformed and the local radius is changed. The method of treating roll flattening by Trinks and Hitchcock (1935) was often used in the subsequent analyses of rolling.

In hot rolling, the plate thickness is generally large, and thus the stress state cannot be assumed uniform in the thickness direction. Further, the friction in hot rolling is generally high, so sticking or constant friction stress is considered to be suitable. To include the stress distribution in the thickness direction and a friction law other than Coulomb friction, Orowan (1943) proposed a more generalized differential equation than Kármán’s equation. Since the solution of this differential equation cannot be expressed explicitly, Orowan and Pascoe (1946), Bland and Ford (1948) and many others worked to establish approximate mathematical expressions for rolling force, torque, load and necessary power.

In the 1950s in the USA and UK, and in the 1960s in Japan, automation of strip rolling in the steel industry began, and many engineers studied and improved these theories in industry.

7. Upper bound method

7.1. Progress of the upper bound method

The upper bound method provides an approximate forming load which is never lower than the correct value. Because of this, the load calculated by this method is safe in selecting forming machines and designing tools, and thus this method has been used practically.

From the relation between velocity and strain rate, the associated strain rate distribution can be determined in the deforming region. With a kinematically admissible velocity field, which satisfies the condition of volume constancy and the velocity boundary condition, together with the flow stress value of the material, an energy dissipation rate and a forming load greater than or equal to the correct values are obtained. This is guaranteed by the limit theorem for rigid-plastic material.

The upper bound theorem came to be known when it was introduced in the book by Hill (1950) together with other bounding theorems. Hill states that Markov wrote a paper about the case of rigid–perfectly plastic material in 1947 in the Russian language.

Let us consider a simple plane-strain case in which a rigid–perfectly plastic body with a shearing flow stress $k$ is deforming by external force $T$ due to a tool moving with velocity $v$. A kinematically admissible velocity field only with velocity discontinuity lines $S^*_d$ with sliding $v^*$ is assumed, where (*) means kinematically admissible field. The upper bound theorem states:

$$Tv \leq \sum \int_{S^*_d} k |v^*| dS_d$$

(17)

where the left side is the correct working rate and the right side is the energy dissipation rate for the plastic deformation along the velocity discontinuities. This inequality means that the energy dissipation rate of the right side is greater or equal to the correct value of the left side. The upper bound value of the forming load $T$ is obtained by dividing the calculated value of the right side by $v$.

Green (1951) applied this theorem to plane-strain compression between smooth plates, and compared the result with that of the slip-line field method as shown in Fig. 38. While a slip-line field requires a long time to draw, a kinematically admissible field with only velocity discontinuity lines $S^*_d$ with sliding $v^*$ is assumed, where (*) means kinematically admissible field. The upper bound theorem states:

Green solved the problem of sheet drawing and bending of notched bars in the early 1950s. From the late 1950s, W. Johnson (1922, working at the University of Cambridge and at UMIST in Manchester) carried out extensive research work by using the upper bound method for plane-strain forging, extrusion, rolling and other forming problems. He optimized the velocity field by assuming proper variables. In the case of extrusion...
shown in Fig. 39 (Johnson and Mellor, 1973), the angle is taken as the parameter for optimization.

Kudo (1960) proposed a method for applying the upper bound method to axi-symmetric forging and extrusion. He divided the axi-symmetric billet into several hypothetical units, and derived mathematical expressions for the velocity of each unit by satisfying the condition of volume constancy and the requirement that surface velocities be consistent with those of neighbouring units and the tool surfaces as demonstrated in Fig. 40.

In the 1960s, when cold forging of steel was increasingly used in the automotive industry, prediction of forming pressure became a very important subject to avoid fracture of the expensive tools. The upper bound method for axi-symmetric deformation appeared just in time and was used extensively in the cold forging industry.

Kudo showed various examples of the use of his method in axi-symmetric forging and extrusion in the book written with Johnson (Johnson and Kudo, 1962). In the 1960s and 70s, his method led to much research activity aimed at finding new types of velocity field, for instance by Kobayashi (1964), Avitzur (1968) and many others.

In the 1970s, the upper bound method was expanded to three dimensional problems by Yang and Lee (1978) and others, and then it was combined with the finite element method, and grew up as the rigid-plastic finite element method as will be explained in the later chapter.

7.2. H. Kudo

Hideaki Kudo (1924–2001) (Fig. 41) graduated from Tokyo Imperial University in 1945 just after World War II, and started...
his career at the Institute of Science and Technology in Tokyo University under the guidance of S. Fukui. He developed axi-symmetric analysis as an approximate energy method without knowing that his method was related to the upper bound theorem. He was awarded his doctoral degree for a thesis of the analysis of forging.

From 1959 to 1960, Kudo stayed in Hannover TH with O. Kienzle and in Manchester University with W. Johnson and wrote papers as well as a book on the upper bound method.

In 1960, he joined the Government Mechanical Engineering Laboratory in Tokyo as the team leader for cold forging. He worked hard for the promotion of the industrialization of cold forging and published many papers and solved various practical problems.

In 1966, Kudo was appointed to a professorship at Yokohama National University where he held the chair for metal forming until 1989. He carried out fundamental studies of slip-line fields, lubrication, material properties and so on as well as unique forming methods such as tension-aided can extrusion.

Kudo is one of the originators of the Japanese Society for the Technology of Plasticity and was President of JSTP for 1985/6. He also originated the International Conference for Technology of Plasticity (ICTP) and served as the chairman of the first meeting held in Tokyo in 1984.

8. Finite element method

8.1. Elastic–plastic finite element method

The finite element method (FEM) was developed for elastic analysis of airplane structures in the 1950s. In this method, a plate was divided into many hypothetical elements, and equations of equilibrium at nodal points were developed. Since the equations were linear in terms of the nodal displacements, they could be solved with the matrix method using digital computers, which had just became useful in some limited research facilities in the USA.

O.C. Zienkiewicz and Y.K. Cheung published a book entitled “The finite element method in structural and continuum mechanics” (Zienkiewicz and Cheung, 1967), in which the method was explained in detail with the software written in the FORTRAN language. Referring to this book, many groups in the world began to develop software using digital computers which had become usable in many countries.

The elastic–plastic FEM was developed as an extension of elastic FEM. Marcal and King (1967) published a paper with an elastic–plastic analysis of a plate specimen with a hole in which the development of a plastic zone was given as shown in Fig. 42. They employed a stepwise computation to follow the deformation. The nodal coordinates and the components of stress and strain in the element were renewed after each step calculation by adding the increments to the values before the step. Next year, Yamada et al. (1968) presented a paper on the stress–strain matrix for elastic–plastic analysis.

This method had a significant impact on researchers in plasticity and soon, many papers on the elastic–plastic analysis of metal forming problems began to appear. Fig. 43 is the result of analysis of the initial state of hydrostatic extrusion by Iwata et al. (1972). In spite of the great potential of the method, it was found that the calculation error accumulated as plastic deformation proceeds.

In this method, the small deformation formulation, the stress value is renewed at the end of each step of the calculation as:

\[ \sigma_x^A = \sigma_x^B + \Delta \sigma_x \]  \hspace{1cm} (18)

where \( \sigma_x^A \) and \( \sigma_x^B \) are the stresses after and before the step computation, and \( \Delta \sigma_x \) is the incremental stress during the step. When an element rotates during the step, the stress state \( \sigma_x^B \) fixed to the element also rotates as shown in Fig. 44. This means that Eq. (18)
cannot hold without modifying $\sigma_0^p$, because $\sigma_0^f$ and $\sigma_0^p$ are defined in different coordinate systems.

To solve this problem, a formulation for large elastic–plastic deformation was put forward by McMeeking and Rice (1975). It is not known well that Kitagawa and Tomita (1974) analyzed a large elastic–plastic deformation problem in a paper published earlier in 1974 due to the paper written in the Japanese language. In the 1980s, commercial software for elastic–plastic deformation appeared, and with the splendid increase of computing speed, the method began to be used in the industry from around 1990. It should be noted that very small time steps are needed even with this formulation to avoid error accumulation, and thus elastic–plastic analysis requires very long computing times even now.

8.2. Rigid-plastic finite element method

Hayes and Marcal (1967) presented a paper on the usage of the FEM for optimizing the upper bound method of a plane stress problem. In the case of the plane stress problem, the stress state could be calculated from the optimized velocity field, but this was not true for other cases.

A normal stress component can be decomposed into a deviatoric stress component $\sigma_x' = \sigma_x - \sigma_m$ and a hydrostatic stress component $\sigma_m = (\sigma_x + \sigma_y + \sigma_z) / 3$. Although the deviatoric stress is related to the strain rate by Eq. (3), the hydrostatic component is not. For example, $\sigma_x$ may be written as:

$$\sigma_x = \sigma_x' + \sigma_m = \frac{2}{3} \frac{\dot{\varepsilon}_x}{\dot{\varepsilon}} + \sigma_m$$

where $\sigma_m$ is left indeterminate with the strain rate associated with the optimized velocity.

If the hydrostatic stress could be determined, this method was expected to have a great potential. Because the stress could be calculated at each step without error accumulation, a drastically shorter computing time than the elastic–plastic FEM was possible although a non-linear problem must be solved by an optimization.

Lee and Kobayashi (1973) and Lung and Mahrenholtz (1973) published papers that enabled stress calculation in the rigid-plastic analysis. Their theoretical basis was the variational principle with a Lagrange multiplier, which had been presented in the book by Washizu (1968). This principle states that when the rigid-plastic problem is optimized with the Lagrange multiplier to handle volume constancy, the multiplier coincides with the hydrostatic stress.

In FEM with the above principle, one or more multipliers are needed for each element to obtain the velocity field which satisfies the incompressibility condition. Since the Lagrange multipliers increase the number of variables, the computation time becomes very long for large scale problems. K. Mori and K. Osakada developed a finite element method allowing for slight volume change without increasing the number of variables. In this method the hydrostatic stress was calculated directly from the slight volume change which did not give significant influence to the deformation. Fig. 45 shows a result of rolling simulated with this method (Mori and Osakada, 1982).

9. Concluding remarks

With the fast development of information technology in the last 20 years, the finite element method has become the main tool of metal forming analysis. To realize more accurate simulation, detailed research is still needed into various areas related to metal forming such as anisotropy development and changes of metalurgical and mechanical properties during deformation, inelastic behaviour in unloading and reloading, lubrication, friction, seizure and fracture.

It seems to be inevitable that the finite element method will be used more directly in industry. Simulation software may be integrated into CAD based systems, in which forming simulation is carried out directly from CAD of forming tools. In order to enable small scale metal forming enterprises to use simulation, low cost software with simplified operations is required.

Although it is impossible for the author to predict far into the future, on-line control of metal forming processes with simultaneous simulation may become possible by the increased computation speed and new computing algorithms.

Once the simulation method is well advanced, metal forming engineers will be able to concentrate on more creative and innovative works, e.g., developments of forming processes for products with very high dimensional accuracy, forming machines for silent environments, tool coatings for dry metal forming and thermomechanical processes with low tool pressure and high product strength, etc.

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